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we obtain as their internal product

$$\begin{aligned} a \mid b &= (\alpha_1 e_1 + \alpha_2 e_2)(\beta_1 e_2 - \beta_2 e_1) \\ &= (\alpha_1 \beta_1 + \alpha_2 \beta_2)\, e_1 e_2 \\ &= \alpha_1 \beta_1 + \alpha_2 \beta_2. \end{aligned}$$

Similarly we find for the internal square

$$a \mid a = \alpha_1^2 + \alpha_2^2.$$

[TO BE CONTINUED.]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.

By MR. MARCUS BAKER, Washington, D. C.

[CONTINUED FROM VOL. I, PAGE 138.]

GROUP II. PART I.

$\Delta =$

32. $\frac{1}{4}\sqrt{8m_c^2(a^2+b^2-2m_c^2)-(a^2-b^2)^2}$

33. $\frac{1}{2}h_c\left(\sqrt{a^2-h_c^2}+\sqrt{b^2-h_c^2}\right)$

34. $\frac{1}{2}\beta_c ab\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{1-\frac{1}{4}\beta_c^2\left(\frac{1}{a}+\frac{1}{b}\right)^2}$

35. $\frac{1}{3}\sqrt{4m_b^2m_c^2-\left[\frac{9}{4}a^2-(m_b^2+m_c^2)\right]^2}=\frac{2}{3}\sqrt{m_b^2m_c^2-k_a^4}$

where $2k_a^2=\frac{9}{4}a^2-(m_b^2+m_c^2)$, etc.

36. $\dfrac{h_b^3\sqrt{a^2-h_b^2}-h_c^3\sqrt{a^2-h_c^2}}{2\,(h_b^2-h_c^2)}$

37. $\frac{1}{6}h_a\left(\sqrt{4m_b^2-h_a^2}+\sqrt{4m_c^2-h_a^2}\right)$

38. $\frac{1}{2}ah_a.$

39. $\frac{1}{2}\sqrt{abh_ah_b}=\frac{1}{2}\,\dfrac{a\pm b}{\dfrac{1}{h_a}\pm\dfrac{1}{h_b}}$

GROUP II. PART I.—*Continued.*

$\Delta =$

40.* $h_a\sqrt{\left(h_a^2+\sqrt{(m_a^2-h_a^2)(\beta_a^2-h_a^2)}\right)\left(\sqrt{\frac{m_a^2-h_a^2}{\beta_a^2-h_a^2}}-1\right)}$

41. $\frac{1}{2}h_a\left(\sqrt{\frac{1}{4}a^2+m_a^2-h_a^2+a\sqrt{m_a^2-h_a^2}}+\sqrt{\frac{1}{4}a^2+m_a^2-h^2-a\sqrt{m_a^2-h_a^2}}\right)$

$=\frac{1}{2}h_a\left(\sqrt{\frac{1}{4}a^2+k_a^2+ak}+\sqrt{\frac{1}{4}a^2+k_a^2-ak}\right)$, where $k_a = m_a^2-h_a^2$, etc.

42. $\frac{Rh_bh_c}{a}$

43. $\frac{ab}{4R}\sqrt{2(a^2+b^2-2m_c^2)}$

44. $h_a\left(\sqrt{b^2-h_a^2}\pm\sqrt{m_a^2-h_a^2}\right)$

45.† $\frac{ab}{8R^2}\left(a\sqrt{4R^2-b^2}+b\sqrt{4R^2-a^2}\right)$

46. $2R^2\left[(f-2x^2)\sqrt{(g-2x^2)(1-g+2x^2)}\right.$

$\left.+(g-2x^2)\sqrt{(f-2x^2)(1-f+2x^2)}\right]$,

where $f=\frac{m_a^2+2m_b^2}{3R^2}$, $g=\frac{2m_a^2+m_b^2}{3R^2}$, and x ($=\sin C$) is to be found from the equation

$$x=\sqrt{(f-2x^2)(1-g+2x^2)}+\sqrt{(g-2x^2)(1-f+2x^2)}.$$

47. $\frac{sh_ar_a}{2(h_a+r_a)}$

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*If μ and ϕ are auxiliaries determined from the equations $\cos\mu=\frac{h_a}{m_a}$ and $\cos\phi=\frac{h_a}{\beta_a}$, then

$$\Delta=\frac{h_a^2}{\cos\mu}\sqrt{\frac{\sin 2(\mu-\phi)}{\sin 2\phi}}.$$

[For this elegant expression I am indebted to my friend Mr. Charles H. Kummell, of the U. S. Coast and Geodetic Survey.]

† $\sqrt{R^2h_a^2-\Delta^2}+\sqrt{R^2h_b^2-\Delta^2}=\frac{1}{2}\left[\frac{Rh_ah_b}{\Delta}\right]^2$

GROUP II. PART II.

$\Delta =$

48. $\frac{1}{2}ab \sin C$

49. $\frac{1}{2}\frac{h_a h_b}{\sin C}$

50. $\frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} = \frac{\frac{1}{2}a^2}{\cot B + \cot C}$

51. $\frac{1}{2}h_a{}^2 \frac{\sin A}{\sin B \sin C} = \frac{1}{2}h_a{}^2 (\cot B + \cot C) = \frac{1}{2}h_a{}^2 (\sin 2B + \sin 2C)$

52. $\frac{1}{2}(a^2 - b^2) \frac{\sin A \sin B}{\sin (A - B)} = \frac{1}{4}(a^2 \sin 2B + b^2 \sin 2A) = \frac{\frac{1}{2}(a^2 - b^2)}{\cot A - \cot B}.$

53. $\frac{1}{2}(h_b{}^2 - h_a{}^2) \frac{\sin A \sin B}{\sin (A - B)} \cdot \frac{1}{\sin^2 C} = \frac{1}{2}h_a h_b \frac{\tan \frac{1}{2}A \tan B \tan \frac{1}{2}C}{\cos A}$

$= h_a h_b \frac{\tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C}{\cos A (1 - \tan^2 \frac{1}{2}B)}$

54. $\frac{2m_a{}^2 \sin A \sin B \sin C}{2(\sin^2 A + \sin^2 B + \sin^2 C) - 3 \sin^2 A}$

55. $\frac{2(m_b{}^2 - m_a{}^2)}{3(\cot A - \cot B)}$

56. $\frac{1}{3}\sqrt{4m_b{}^2 m_c{}^2 - [9R^2 \sin^2 A - (m_b{}^2 + m_c{}^2)]^2}$

57. $\frac{2R^4}{m_a{}^2} \sin A \sin B \sin C (-\sin^2 A + 2 \sin^2 B + 2 \sin^2 C)$

58. $\frac{4m_a{}^2 + 3a^2}{8(\cot A + \cot B + \cot C)}$

59. $\frac{4(m_a{}^2 + m_b{}^2) - 3c^2}{4(\cot A + \cot B + \cot C)}$

60. $\frac{1}{2}\beta_a{}^2 \frac{\sin A}{\sin B \sin C} \sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B)$

61. $Rh_a \sin A$

62. $\frac{1}{2}a\beta_a \sin (C + \frac{1}{2}A)$

63. $R^2 \sin 2A (1 + \cos C)$

64. $Ra \sin B \sin C = Ra (\cos A + \cos B \cos C)$

65. $Rh_a \tan A \sin B \tan \frac{1}{2}C$

66. $R\beta_a \tan A \sin B \tan \frac{1}{2}C \sin (C + \frac{1}{2}A)$

GROUP II. PART II.—*Continued.*

$\Delta =$

67. $$\frac{h_a(h_b+h_c)}{4\sin(C+\tfrac{1}{2}A)\cos\tfrac{1}{2}A}$$

68. $$\frac{\beta_a(h_b+h_c)}{4\cos\tfrac{1}{2}A}$$

69. $$\tfrac{1}{4}\beta_a^2(b+c)\left(\frac{1}{b}+\frac{1}{c}\right)\tan\tfrac{1}{2}A.$$

70. $$\tfrac{1}{2}\beta_a\beta_b\frac{\sin(B+\tfrac{1}{2}A)\sin(A+\tfrac{1}{2}B)}{\sin C}$$

71. $$\beta_a\beta_b\beta_c\frac{-\beta_a\sin\tfrac{1}{2}A+\beta_b\sin\tfrac{1}{2}B+\beta_c\sin\tfrac{1}{2}C}{-\beta_b\beta_c\cos\tfrac{1}{2}A+\beta_c\beta_a\cos\tfrac{1}{2}B+\beta_a\beta_b\cos\tfrac{1}{2}C}$$

72. $$\tfrac{1}{2}a\sin B\left(a\cos B+\sqrt{b^2-a^2\sin^2 B}\right)=\tfrac{1}{2}b\sin A\left(b\cos A+\sqrt{a^2-b^2\sin^2 A}\right)$$

GROUP III. PART I.

73. $$sr=(s-a)r_a$$

74. $$\tfrac{1}{3}r\left(\sqrt{2m_a^2+2m_b^2-m_c^2}+\sqrt{2m_b^2+2m_c^2-m_a^2}+\sqrt{2m_c^2+2m_a^2-m_b^2}\right)$$
$$=\tfrac{1}{3}r_a\left(\sqrt{2m_a^2+2m_b^2-m_c^2}-\sqrt{2m_b^2+2m_c^2-m_a^2}+\sqrt{2m_c^2+2m_a^2-m_b^2}\right)$$

75. $$\sqrt{\frac{r r_b r_c}{-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}}=\sqrt{\frac{r_a r_b r_c}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}}$$

76. $$\frac{R}{2s}(h_ah_b+h_bh_c+h_ch_a)=\frac{R}{2(s-a)}(h_ah_b-h_bh_c+h_ch_a)$$

77. $$\sqrt{\tfrac{1}{2}Rr(h_ah_b+h_bh_c+h_ch_a)}=\sqrt{\tfrac{1}{2}Rr_a(h_ah_b-h_bh_c+h_ch_a)}$$

78. $$\frac{s}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=\frac{s-a}{-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}$$

79. $$\sqrt{\frac{r}{\left(-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$$
$$=\sqrt{\frac{r_a}{\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}-\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$$

GROUP III. PART II.

$\Delta =$

80. $a(-\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$

$= 2s(\beta_a \sin \tfrac{1}{2}A + \beta_b \sin \tfrac{1}{2}B + \beta_c \sin \tfrac{1}{2}C)$

81. $\sqrt{2r\beta_a\beta_b\beta_c \left[\dfrac{\cos(A-B) + \cos(B-C) + \cos(C-A) + 1}{\cos A + \cos B + \cos C - 1}\right]}$

$= \sqrt{2r_a\beta_a\beta_b\beta_c \left[\dfrac{\cos(A-B) + \cos(B-C) + \cos(C-A) + 1}{-\cos A + \cos B + \cos C + 1}\right]}$

82. $s^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = (s-a)^2 \tan \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C$

83. $r^2 \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C = r_a^2 \cot \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C.$

GROUP IV. PART I.

84. $a\dfrac{rr_a}{r_a - r} = a\dfrac{r_br_c}{r_b + r_c}$

85. $rr_a\dfrac{r_b + r_c}{a} = r_br_c\dfrac{r_a - r}{a}$

86. $(a+b)\dfrac{rr_c}{r + r_c} = (a-b)\dfrac{r_ar_b}{r_a - r_b}$

87. $rr_a\dfrac{r_b - r_c}{b - c} = r_br_c\dfrac{r + r_a}{b + c}$

88. $rr_a\sqrt{\dfrac{r_b + r_c}{r_a - r}} = r_br_c\sqrt{\dfrac{r_a - r}{r_b + r_c}}$

89. $rr_a\sqrt{\dfrac{4R - (r_a - r)}{r_a - r}} = r_br_c\sqrt{\dfrac{4R - (r_b + r_c)}{r_b + r_c}}$

90. $\dfrac{sh_ar_a}{h_a + 2r_a} = \dfrac{(s-a)h_ar}{h_a - 2r}$

91. $r\sqrt{\dfrac{h_ar_br_c}{h_a - 2r}} = r_a\sqrt{\dfrac{h_ar_br_c}{h_a + 2r_a}}$

92. $\sqrt{\dfrac{rr_a}{\left(\dfrac{1}{h_a} - \dfrac{1}{h_b} + \dfrac{1}{h}\right)\left(\dfrac{1}{h_a} + \dfrac{1}{h_b} - \dfrac{1}{h_c}\right)}}$

$= \sqrt{\dfrac{r_br_c}{\left(\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}\right)\left(-\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}\right)}}$

GROUP IV. PART II.

$\Delta =$

93. $r r_a \cot \frac{1}{2}A = r_b r_c \tan \frac{1}{2}A$

GROUP V.

94. $r^2 \cot \frac{1}{2}A + 2Rr \sin A = r_a^2 \cot \frac{1}{2}A - 2Rr_a \sin A = r_b^2 \cot \frac{1}{2}A - 2Rr_b \sin A$
$= r_c^2 \cot \frac{1}{2}A - 2Rr_c \sin A.$

MISCELLANEOUS EXPRESSIONS FOR THE AREA OF A PLANE TRIANGLE.

If we designate the distances from the orthocentre to the sides of the triangle by k_a, k_b, k_c and from the orthocentre to the vertices by k_a', k_b', k_c', then

95. $\Delta = \frac{1}{2}(ak_a + bk_b + ck_c)$

96.* $\Delta = \frac{1}{4}(ak_a' + bk_b' + ck_c')$

97.* $\Delta = \frac{1}{8R}(a^2k_a' \operatorname{cosec} A + b^2k_b' \operatorname{cosec} B + c^2k_c' \operatorname{cosec} C).$

If we designate the distances between the centres of the escribed circles by d_a, d_b, d_c, then

98.* $\Delta = \frac{1}{4}\sqrt{2d_ad_bd_cr \sin A \sin B \sin C}$

99. $\Delta = \frac{d_ad_bd_cr}{16R^2}$

100. $\Delta = \frac{d_ad_bd_c}{4R} \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$

101.* $\Delta = r_ar_b\sqrt{\frac{d_c^2}{(r_a + r_b)^2} - 1}.$

If x_ay_a, x_by_b, x_cy_c designate the rectangular co-ordinates of a triangle in a plane, then

102. $\Delta = \frac{1}{2}[y_a(x_c - x_b) + y_b(x_a - x_c) + y_c(x_b - x_a)]$

and, in polar co-ordinates,

103. $\Delta = \frac{1}{2}[r_ar_b \sin(\theta'' - \theta') + r_br_c \sin(\theta''' - \theta'') + r_cr_a \sin(\theta' - \theta''')].$

If the equations of the sides of a triangle are

$$a_ax + b_ay + c_a = 0; \; a_bx + b_by + c_b = 0; \; a_cx + b_cy + c_c = 0,$$

then

104. $\Delta = \frac{1}{2}\left[\frac{[a_a(b_bc_c - b_cc_b) + a_b(b_cc_a - b_ac_c) + a_c(b_ac_b - b_bc_a)]^2}{(a_ab_b - b_aa_b)(a_bb_c - b_ba_c)(a_cb_a - b_ca_a)}\right].$

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* *Tidsskrift for Mathematik.* 8°. Copenhagen, 1884, 5th Series, Vol. I, No. 2, pp. 78–79. *Mathesis.* 8°. Gand, 1884, November, Vol. IV, p. 232.

If the co-ordinates of the vertices of a triangle are x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c, and

$$\begin{aligned} \alpha &= x_a^2 + y_a^2 + z_a^2, & \lambda &= x_b x_c + y_b y_c + z_b z_c, \\ \beta &= x_b^2 + y_b^2 + z_b^2, & \mu &= x_c x_a + y_c y_a + z_c z_a, \\ \gamma &= x_c^2 + y_c^2 + z_c^2; & \nu &= x_a x_b + y_a y_b + z_a z_b; \text{ then} \end{aligned}$$

105.* $$4\Delta^2 = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \alpha & \nu & \mu \\ 1 & \nu & \beta & \lambda \\ 1 & \mu & \lambda & \gamma \end{vmatrix}$$

or, without determinants,

$$\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,$$

where

$$\begin{aligned} a^2 &= (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2, \\ b^2 &= (x_c - x_a)^2 + (y_c - y_a)^2 + (z_c - z_a)^2, \\ c^2 &= (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2, \end{aligned}$$

and if α, β, γ, λ, μ and ν, have the values set down above, then

$$\begin{aligned} a^2 &= \beta + \gamma - 2\lambda, \\ b^2 &= \gamma + \alpha - 2\mu, \\ c^2 &= \alpha + \beta - 2\nu, \end{aligned}$$

and

$$\begin{aligned} \Delta^2 = {} & 2(\alpha + \beta - 2\nu)(\beta + \gamma - 2\lambda) - (\alpha + \beta - 2\nu)^2 \\ & + 2(\beta + \gamma - 2\lambda)(\gamma + \alpha - 2\mu) - (\beta + \gamma - 2\lambda)^2 \\ & + 2(\gamma + \alpha - 2\mu)(\alpha + \beta - 2\nu) - (\gamma + \alpha - 2\mu)^2, \end{aligned}$$

which reduces to

$$\Delta^2 = 4(\lambda + \mu + \nu)^2 - 8(\alpha\lambda + \beta\mu + \gamma\nu) + 4(\alpha\beta + \beta\gamma + \gamma\alpha).$$

106. $\Delta = \frac{1}{8R}\left[(s - a)D_a^2 + (s - b)D_b^2 + (s - c)D_c^2 - sD^2\right]$ where D, D_a, D_b, D_c are the distances between the circumcentre and the in- and e-centres.

107. $\Delta = 4K$, where K is the area of the triangle formed by joining the feet of the medians.

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* *Giornale di Matematiche*, Naples, 1884, July, Vol. XXII, pp. 207–8.

108. $\Delta = \frac{L}{2\cos A\cos B\cos C}$, where L is the area of the pedal triangle or triangle formed by joining the feet of the perpendiculars.

109. $\Delta = M\frac{(a+b)(b+c)(c+a)}{2abc}$, where M is the area of the triangle formed by joining the feet of the internal bisectors.

110. $\Delta = \frac{2R}{r}\,.\,N$, where N is the area of the triangle formed by joining the points of tangency of the in-circle.

Concluding Note.—Owing to some changes and additions in the foregoing article made since its original preparation and while in press, the statement on page 135, Vol. I, that "the total number of formulæ * * * in this collection is *ninety-three*," etc., is no longer true.

The number of formulæ in the collection is shown below where the second column shows the number that result from counting as distinct those formulæ arising from permutation.

Group I,	31	31
Group II,	41	123
Group III,	11	44
Group IV,	10	60
Group V,	1	12
Miscellaneous Group,	16	18
Total,	110	288

[In the original manuscript all the forms arising from permuting the letters were given in full. To save space in printing, those forms which arose from mere cyclical permutation were omitted.—*O. S.*]

PROBLEM AND SOLUTION.

By Dr. J. E. Hendricks, Des Moines, Iowa.

Problem—To inscribe a rectangle whose sides are c,x in another rectangle whose sides are a,b, and express x in functions of a, b, and c.